MATH 2040 Lecture 21 (24/11/2016)

Setting: $T : V \to V$ linear on a vector space V

(assume $\mathbb{F} = \mathbb{C}$, $\dim V = n$)

$$K_\lambda := \{ v \in V \mid (T - \lambda I)^p v = 0 \text{ for some } p \geq 1 \}$$

Last time: (0) λ eigenvalue, and $E_\lambda \subseteq K_\lambda$

(1) K_λ T-inv. subspace

(2) $\forall \mu \neq \lambda$, $T - \mu I : K_\lambda \xrightarrow{\cong} K_\lambda$

$$(\Rightarrow (T - \mu I)^m : K_\lambda \xrightarrow{\cong} K_\lambda \quad \forall m)$$

(3) $\dim K_\lambda \leq m$, $N(T - \lambda I)^m = K_\lambda$

where m = multiplicity of λ

Theorem: $\boxed{V = K_{\lambda_1} \oplus \cdots \oplus K_{\lambda_k}}$ $(*)$

Proof: ✓(1) $V = K_{\lambda_1} + \cdots + K_{\lambda_k}$

(2) "Direct": If $v_i \in K_{\lambda_i}$ s.t. $v_1 + v_2 + \cdots + v_k = 0$

then each $v_i = 0$ for $i = 1, \dots, k$.

- Prove (1) by induction on k = # of eigenvalues of T

$k = 1$: Only 1 eigenvalue λ

$\Rightarrow$ char. poly. $= f(t) = (-1)^n (t - \lambda)^n$

Cayley-Hamilton $\Rightarrow$ $(T - \lambda I)^n = 0$

$\Rightarrow K_\lambda = N(T - \lambda I)^n = N(0) = V$

Assume it is true for $k-1$ eigenvalues.

Let's say $\exists$ k distinct e.values $\lambda_1, \lambda_2, \dots, \lambda_k$

w/ multiplicities $m_1, m_2, \dots, m_k$

$\Rightarrow$ char. poly $= f(t) = (-1)^n (t-\lambda_1)^{m_1} \cdots (t-\lambda_k)^{m_k}$

Q: To find a subspace $W \subseteq V$ s.t.

$T|_W : W \to W$ has $k-1$ eigenvalues?

$$J = \begin{pmatrix} \overbrace{\boxed{\lambda_1^{\neq 0}}}^{W} & & & \\ & \boxed{\lambda_2^{\neq 0}} & & \\ & & \ddots & \\ & & & \boxed{0} \end{pmatrix}$$

$W = \text{range}(J)$

$= R(T)$

make this 0

block corr. to λ_k :
$$\left[\underbrace{\begin{pmatrix} \lambda_k & 1 & & \\ & \lambda_k & 1 & \\ & & \ddots & 1 \\ & & & \lambda_k \end{pmatrix}}_{\leq m_k} - \lambda_k I \right]^{m_k} = \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix}^{m_k} = 0$$

Define: $W = R\left((T-\lambda_k I)^{m_k}\right)$

Check: ✓. W T-invariant ($\because$ $(T-\lambda_k I)$-inv $\Rightarrow$ T-inv.)

✓. $K_{\lambda_i} \subseteq W$ for $i = 1, \dots, k-1$

($\because$ $(T-\lambda_k I)^{m_k} : K_{\lambda_i} \xrightarrow{\cong} K_{\lambda_i} \Rightarrow K_{\lambda_i} \subseteq W$)

Restrict T onto W

$$T\big|_W : W \to W \quad \text{linear}$$

Claim: This has $k-1$ distinct eigenvalues. (Ex:)

By induction,

$$W = K_{\lambda_1} + \cdots + K_{\lambda_{k-1}}$$

Claim: $V = (K_{\lambda_1} + \cdots + K_{\lambda_{k-1}}) + K_{\lambda_k}$

Pf: Take any $v \in V$,

$$W \ni (T - \lambda_k I)^{m_k} v = \underset{\substack{\cap \\ K_{\lambda_1}}}{v_1} + \underset{\substack{\cap \\ K_{\lambda_2}}}{v_2} + \cdots + \underset{\substack{\cap \\ K_{\lambda_{k-1}}}}{v_{k-1}}$$

$$= (T - \lambda_k I)^{m_k} \underset{\substack{\cap \\ K_{\lambda_1}}}{u_1} + \cdots + (T - \lambda_k I)^{m_k} \underset{\substack{\cap \\ K_{\lambda_{k-1}}}}{u_{k-1}}$$

$$\Rightarrow (T - \lambda_k I)^{m_k} \underbrace{(v - u_1 - \cdots - u_{k-1})}_{u_k \in K_{\lambda_k}} = 0$$

$$\Rightarrow \quad v = u_1 + \cdots + u_{k-1} + u_k \quad \text{where } u_i \in K_{\lambda_i}.$$

Proof of (2): "Direct"

Suppose $\quad v_1 + v_2 + \cdots + v_k = 0 \quad$ where $v_i \in K_{\lambda_i}$

$$(T-\lambda_k I)^{m_k} v_1 + (T-\lambda_k I)^{m_k} v_2 + \cdots + \underbrace{(T-\lambda_k I)^{m_k} v_k}_{=0} = 0$$

Lemma (2)

By induction, $v_1 = v_2 = \cdots = v_{k-1} = 0$

$$\Rightarrow \quad v_k = 0 .$$